A Bayesian Approach to the Laplacian-like Inverse Problem

Sergios Agapiou Stig Larsson Andrew Stuart

The University of Warwick, Coventry, UK

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Summary

Inverse Problems are concerned with determining causes for a desired or an observed effect. In our context, we want to invert a linear operator from noisy observations of its image.

Aim: To study the consistency of the Bayesian Approach to Linear Inverse Problems in the small noise limit.

We start with the Laplacian-like Inverse Problem with noise and prior covariances diagonalizable in the eigenbase of the Laplacian and generalize to more general situations. We are always assuming a separable Hilbert-Space setting.

Numerical results are presented which support this theory.

Problem set-up - General Case

Consider the Inverse Problem to find u from y, for

$$y = \mathcal{A}^{-\ell}u + \frac{1}{\sqrt{n}}\xi, \ \ell > 0, \qquad (IP)$$

where $\xi \sim \mathcal{N}(0, C_1)$ and $\mathcal{A}$ is selfadjoint.

- Likelihood: for fixed u the law of $y|u$, called the *likelihood*, is $\mathcal{N}(\mathcal{A}^{-\ell}u, C_1)$.

- Prior: we now need to choose a *prior* distribution for the unknown u, based on any prior knowledge we have about u. Let $u \sim \mathcal{N}(0, \tau^2 C_0)$.

- Posterior: in the Bayesian Approach, the solution of (IP) is the distribution of $u|y$, called the *posterior* distribution. Here the posterior is $\mathcal{N}(m, C)$ where

$$m = C_0\mathcal{A}^{-\ell}(n^{-1}\tau^{-2}C_1 + \mathcal{A}^{-\ell}C_0\mathcal{A}^{-\ell})^{-1}y := Ly$$

and

$$C = \tau^2 C_0 - \tau^2 L\mathcal{A}^{-\ell}C_0.$$

Assume we have observations of the form

$$y_n = \mathcal{A}^{-\ell}u^\dagger + \frac{1}{\sqrt{n}}\xi \qquad (DM)$$

where $u^\dagger$ is the true solution. Then the posterior mean is

$$m_n = Ly_n. \qquad (PM)$$

We examine the consistency of the posterior mean in the sense

$$\mathbb{E}\left|\left|m_n - u^\dagger\right|\right|^2 \to 0$$

as $n \to \infty$ and for appropriate choice of $\tau = \tau(n)$.

Problem set-up - Diagonal Case

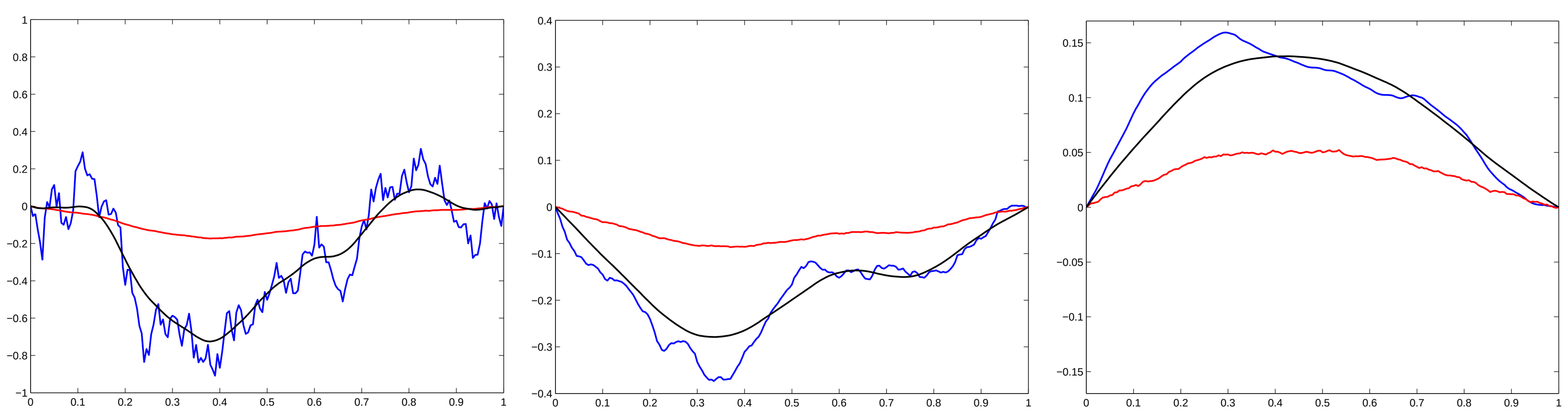
Let $\mathbf{\Delta}$ be the Dirichlet Laplacian in $I = (0, 1)$ and $\{k^2, \phi_k\}$ its orthonormal eigenbasis. Define the "Sobolev-like" spaces

$$\mathcal{H}^s = \{u : \sum_{k=1}^{\infty} k^{2s}\langle u, \phi_k\rangle^2 < \infty\}, \qquad ||u||_s^2 = \sum_{k=1}^{\infty} k^{2s}\langle u, \phi_k\rangle^2.$$

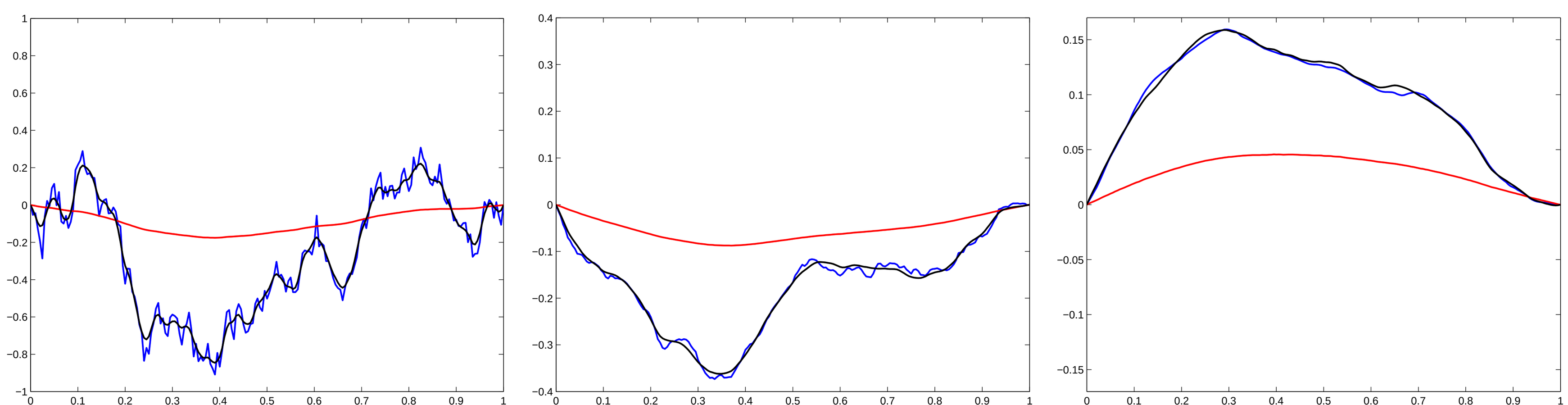
Suppose that in (IP) we have $C_0 = (-\mathbf{\Delta})^{-\frac{1}{2}-\alpha}$, $\mathcal{A}^{-\ell} = C_0^{\frac{2\ell}{1+2\alpha}}$ and $C_1 = C_0^{\frac{2\beta}{1+2\alpha}}$ where $\alpha > 0, \beta \geq 0$, i.e. C_0, $\mathcal{A}^{-\ell}$ and C_1 are diagonalizable in the same eigenbasis.

Following the usual practice in Inverse Problems, we will assume in (DM) that we have the a-priori information on the regularity of $u^\dagger$, that $u^\dagger \in \mathcal{H}^\gamma, \ \gamma \geq 0$.

Numerical Simulation - Diagonal Case



(a) $\gamma = \frac{1}{2}$, $n = 10^4$, $||m - u^\dagger|| \approx 0.29$ (b) $\gamma = 1$, $n = 10^4$, $||m - u^\dagger|| \approx 0.097$ (c) $\gamma = \frac{3}{2}$, $n = 10^4$, $||m - u^\dagger|| \approx 0.046$



(d) $\gamma = \frac{1}{2}$, $n = 10^6$, $||m - u^\dagger|| \approx 0.11$ (e) $\gamma = 1$, $n = 10^6$, $||m - u^\dagger|| \approx 0.022$ (f) $\gamma = \frac{3}{2}$, $n = 10^6$, $||m - u^\dagger|| \approx 0.0073$

Figure: Realizations of the true solution (blue), the posterior mean (black) and the observation (red) for $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\ell = \frac{1}{2}$, for three values of γ and for $n = 10^4$ (1st line) and $n = 10^6$ (2nd line).

Observation: We have a better reconstruction when the truth is more regular.

Posterior Consistency - Diagonal Case

Define $\Delta = \frac{1}{4} + \frac{\alpha}{2} + \ell - \frac{\beta}{2}$, and assume that $\Delta > 0$ (the prior is indeed regularizing).

We study optimal rates of convergence for the different values of γ:

-**Theorem:** If the truth is smooth enough, in particular if $\gamma \geq 0$

i) If the noise and the true solution are not too smooth, $\beta - 2\ell \leq \frac{1}{2}$ and $\gamma < 4\Delta$, for

$$\tau_n = n^{\frac{\alpha-\gamma}{4\Delta+2\gamma-2\alpha+\varepsilon}},$$

we have

$$\mathbb{E}\left|\left|m_n - u^\dagger\right|\right|^2 \asymp n^{\frac{-\gamma}{2\Delta+\gamma-\alpha+\varepsilon}}.$$

If the true solution is too smooth, $\gamma \geq 4\Delta$, then the rate does not improve as γ increases, i.e. the rate saturates at

$$n^{\frac{-4\Delta}{6\Delta-\alpha+\varepsilon}}.$$

ii) If the noise is too smooth, $\beta - 2\ell > \frac{1}{2}$, then the resulting rate is arbitrarily close to n^{-1}.

Observation: in (i) for $\alpha > \gamma$ the prior distribution becomes uniform as $n \to \infty$, $\alpha < \gamma$ it contracts to a Dirac, $\alpha = \gamma$ it stays fixed. In the last case the prior regularity matches the regularity of the true solution so rescaling is redundant.

Simulated Rates of Convergence

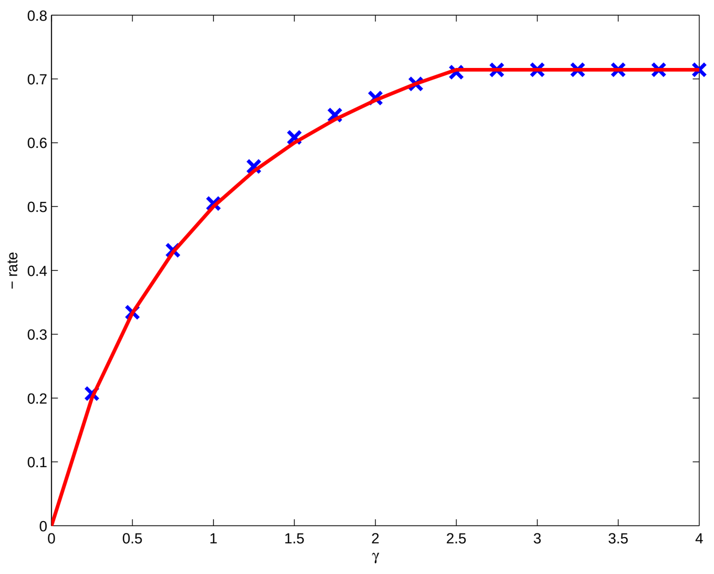


Figure: Rates of convergence of $\mathbb{E}\left|\left|m_n - u^\dagger\right|\right|^2$ plotted against γ. In red we have the rates predicted by the theory and in blue we have estimated rates using the average of 160 realizations.

Observation: The estimated rates saturate at $\gamma = 2.5$ as predicted by the theory.

Posterior Consistency - Non-Diagonal Case

-**Theorem:** Under weaker, non-diagonal assumptions on the equivalence of C_0 to $\mathcal{A}^{-\ell}$ and C_1, using PDE methods and interpolation inequalities we have that for $\theta \in [0, 1]$

$$\mathbb{E}\left|\left|C_0^{\frac{2\ell-\beta-2\theta\Delta}{1+2\alpha}}(m_n - u^\dagger)\right|\right|^2 \leq c\mathbb{E}(\kappa^2)n^{\frac{\theta_2+\theta-2}{\theta_1-\theta_2+2}},$$

where

$$\kappa = \max\left\{\left|\left|C_0^{\frac{2\theta_1\Delta-\beta}{1+2\alpha}}\xi\right|\right|, \left|\left|C_0^{\frac{2(\theta_2-1)\Delta}{1+2\alpha}-\frac{1}{2}}u^\dagger\right|\right|\right\}$$

and where θ_1, θ_2 are chosen so that $\mathbb{E}(\kappa^2) < \infty$.

In the diagonal case the assumptions are trivially satisfied, so we can examine if the general theory agrees with the rates obtained in the diagonal case.

- For $\beta - 2\ell \leq 0$, we can choose $\theta = \frac{2\ell-\beta}{2\Delta}$ in the general method to get the rate for $\mathbb{E}\left|\left|m_n - u^\dagger\right|\right|^2$. Compared to the diagonal case, we require higher regularity of the true solution in order to have convergence ($\gamma \geq \frac{1}{2} + \alpha$) and the rate saturates earlier ($\gamma = \frac{1}{2} + \alpha + 2\Delta$). For $\gamma \in \left[\frac{1}{2} + \alpha, \frac{1}{2} + \alpha + 2\Delta\right]$ the rate agrees with the diagonal case.

The discrepancies can be explained by the fact that in the general method, the choice of θ_1, θ_2 which determines both the minimum requirement on the regularity of $u^\dagger$ and the saturation point, is independent of the choice of θ. This means that on the one hand to get convergence of $\mathbb{E}\left|\left|m_n - u^\dagger\right|\right|^2$ we require conditions which secure the convergence of stronger norms of the error and on the other hand the saturation rate for $\mathbb{E}\left|\left|m_n - u^\dagger\right|\right|^2$ is the same as the saturation rates for weaker norms of the error.

- For $\beta - 2\ell > 0$, the general method provides rates only for stronger norms of the error, hence it secures the convergence of $\mathbb{E}\left|\left|m_n - u^\dagger\right|\right|^2$ but at suboptimal rates.

References

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