Regularization of Linear Inverse Problems by Bayesian Methodology

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Inverse Problems

- Inverse Problems are concerned with determining causes for a desired or an observed effect.
- $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ known self-adjoint, positive definite linear operator with bounded inverse, $\mathcal{X}$ separable Hilbert space.
- Consider the inverse problem to find u from y, where y noisy observation of $\mathcal{A}^{-1}u$.
- Model:

$$y = \mathcal{A}^{-1}u + \frac{1}{\sqrt{n}}\xi, \tag{1}$$

where $\frac{1}{\sqrt{n}}\xi$ an additive noise.

Inverse Problems are Ill-Posed - Deterministic Approach

- Problem (1) ill-posed:
  - Existence of solution not guaranteed
  - Solutions do not depend continuously on the data
- Tikhonov-Phillips Regularization: u approximated by minimizer of

$$J_0(u) := \frac{1}{2}\left\|\mathcal{C}_1^{-\frac{1}{2}}(y - \mathcal{A}^{-1}u)\right\|^2 + \frac{\lambda}{2}\left\|\mathcal{C}_0^{-\frac{1}{2}}u\right\|^2,$$

$\mathcal{C}_i : \mathcal{X} \to \mathcal{X}$, self-adjoint, possibly compact, positive definite linear operators.

- λ regularization parameter, appropriate function of noise level $n^{-\frac{1}{2}}$ which shrinks to zero as $n \to \infty$ to recover unknown u.

Bayesian Approach

Bayesian framework: assume $\xi \sim \mathcal{N}(0, \mathcal{C}_1)$, $\mathcal{C}_1 : \mathcal{X} \to \mathcal{X}$ selfadjoint positive definite.

- Likelihood: for fixed u, $y|u \sim \mathcal{N}(\mathcal{A}^{-1}u, \frac{1}{n}\mathcal{C}_1)$.
- Prior: choose *prior* distribution for unknown u, encoding prior knowledge. Let $u \sim \mathcal{N}(0, \tau^2\mathcal{C}_0)$, $\mathcal{C}_0 : \mathcal{X} \to \mathcal{X}$ selfadjoint positive definite trace class.
- Posterior: in the Bayesian Approach, the solution of problem (1) is the distribution of $u|y$, called the *posterior* distribution.
- Link: Bayes rule

$$\mathbb{P}(u|y) \propto \mathbb{P}(y|u)\mathbb{P}(u)$$

Bayesian Approach - link to Tikhonov

- Work in infinite dimensional setting where we can show

$$\mathbb{P}(u|y) = \mathcal{N}(m, \frac{1}{n}\mathcal{B}_\lambda^{-1})$$

$$\mathcal{B}_\lambda = \mathcal{A}^{-1}\mathcal{C}_1^{-1}\mathcal{A}^{-1} + \lambda\mathcal{C}_0^{-1}, \quad \lambda = \frac{1}{n\tau^2}$$

$$\mathcal{B}_\lambda m = \mathcal{A}^{-1}\mathcal{C}_1^{-1}y.$$

- $\mathcal{B}_\lambda$ depends on n and τ only through λ.
- Observation: m minimizer of J_0; posterior mean is Tikhonov-Phillips solution of (1)!

Setting - Assumptions

- Hilbert Scale $(X^s)_{s\in\mathbb{R}}$, for $X^s = \mathcal{D}(\mathcal{C}_0^{-\frac{s}{2}})$ with $\langle u, v\rangle_s = \left\langle \mathcal{C}_0^{-\frac{s}{2}} u, \mathcal{C}_0^{-\frac{s}{2}} v \right\rangle$.

- Assumptions on $\mathcal{A}, \mathcal{C}_0$ and $\mathcal{C}_1$

$$\mathcal{C}_1 \simeq \mathcal{C}_0^{\beta}, \quad \beta \geq 0$$

$$\mathcal{A}^{-1} \simeq \mathcal{C}_0^{\ell}, \quad \ell > 0.$$

- We have

$$\mathcal{B}_\lambda = \mathcal{A}^{-1}\mathcal{C}_1^{-1}\mathcal{A}^{-1} + \lambda\mathcal{C}_0^{-1} \simeq \mathcal{C}_0^{2\ell-\beta} + \lambda\mathcal{C}_0^{-1}.$$

Assume $\Delta := 2\ell - \beta + 1 > 0$, i.e. prior is regularizing.

Bayesian Approach - Posterior Consistency

- Assume observations

$$y^\dagger = \mathcal{A}^{-1}u^\dagger + \frac{1}{\sqrt{n}}\xi, \quad \xi \sim \mathcal{N}(0, \mathcal{C}_1)$$

where $u^\dagger \in \mathcal{X}$ is the fixed true solution.

- This data model gives $\mathbb{P}(u|y = y^\dagger) = \mathcal{N}(m_\lambda^\dagger, \frac{1}{n}\mathcal{B}_\lambda^{-1})$, where

$$\mathcal{B}_\lambda m_\lambda^\dagger = \mathcal{A}^{-1}\mathcal{C}_1^{-1}y^\dagger.$$

Posterior Consistency - Error Equation

$$\mathcal{B}_\lambda m_\lambda^\dagger = \mathcal{A}^{-1}\mathcal{C}_1^{-1}y^\dagger = \underbrace{\mathcal{A}^{-1}\mathcal{C}_1^{-1}\mathcal{A}^{-1}u^\dagger} + \frac{1}{\sqrt{n}}\mathcal{A}^{-1}\mathcal{C}_1^{-1}\xi$$

$$\mathcal{B}_\lambda u^\dagger = \underbrace{\mathcal{A}^{-1}\mathcal{C}_1^{-1}\mathcal{A}^{-1}u^\dagger} + \lambda\mathcal{C}_0^{-1}u^\dagger.$$

Set $e = m_\lambda^\dagger - u^\dagger$

$$\mathcal{B}_\lambda e = \frac{1}{\sqrt{n}}\mathcal{A}^{-1}\mathcal{C}_1^{-1}\xi - \lambda\mathcal{C}_0^{-1}u^\dagger. \qquad \text{(err)}$$

- Assume $u^\dagger \in X^\gamma$. Want rates

$$\mathbb{E}\,\|e\|_\eta^2 = \mathcal{O}(f(n,\eta,\gamma,\ell,\beta)).$$

Main result

- Set $\eta_i = (1-\theta_i)(\beta - 2\ell) + \theta_i$, where $\theta_i \in [0,1]$.

  Let $\kappa^2 = \max\{\|\xi\|^2_{2\beta-2\ell-\eta_1}, \|u^\dagger\|^2_{2-\eta_2}\}$ and choose θ_1, θ_2 s.t. $\mathbb{E}(\kappa^2) < +\infty$.

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Theorem (Agapiou, Larsson, Stuart)

For any $\theta \in [0,1]$,

$$\mathbb{E}\,\|e\|^2_\eta \leq c\mathbb{E}(\kappa^2) n^{\frac{2-\theta_2-\theta}{\theta_2-\theta_1-2}}$$

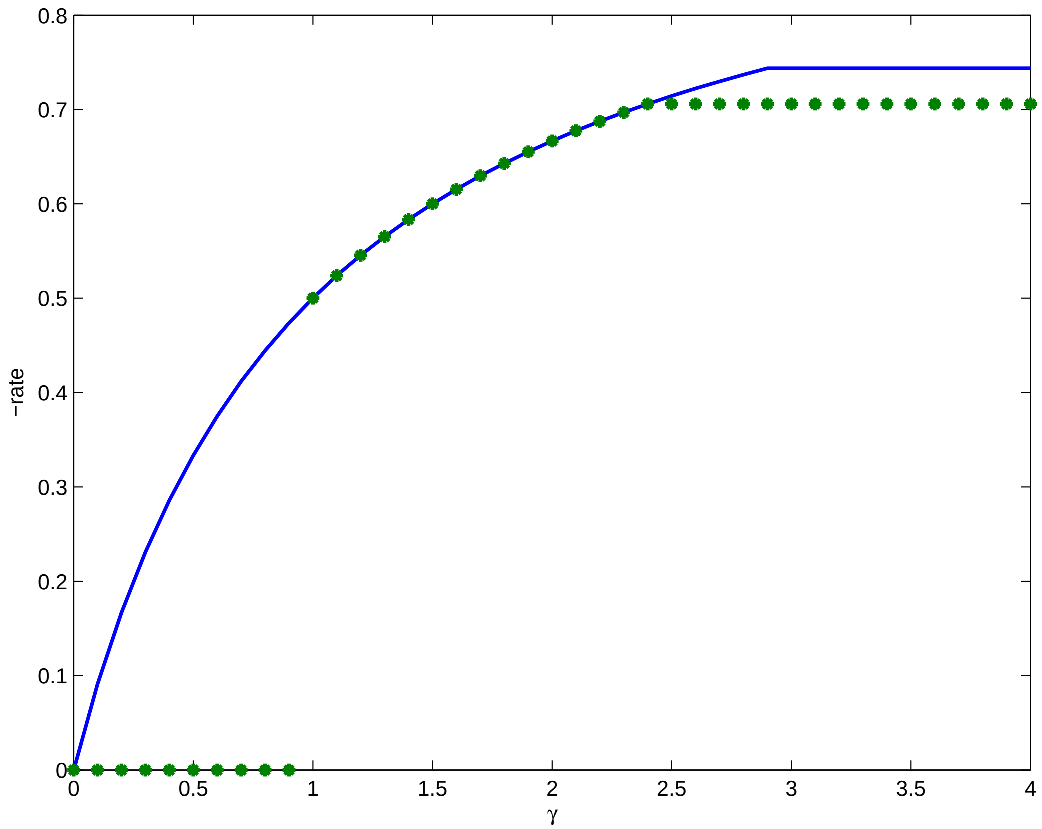
where $\eta = (1-\theta)(\beta - 2\ell) + \theta$.

Optimality

Diagonal Case: $\mathcal{A}^{-1} = \mathcal{C}_0^{\ell}$ and $\mathcal{C}_1 = \mathcal{C}_0^{\beta}$ gives sharp rates.

Our assumptions satisfied trivially.

- Assume $\mathcal{C}_0 \asymp \text{diag}\{k^{-2}\}$, $u^{\dagger} \in X^{\gamma}$. Compare rates of convergence for $\ell = \beta = 1/2$.



Proof - Making Sense of the error equation

- Define the bilinear form $B : X^1 \times X^1 \to \mathbb{R}$

$$B(u,v) = \left\langle \mathcal{C}_1^{-\frac{1}{2}} \mathcal{A}^{-1} u, \mathcal{C}_1^{-\frac{1}{2}} \mathcal{A}^{-1} v \right\rangle + \lambda \left\langle \mathcal{C}_0^{-\frac{1}{2}} u, \mathcal{C}_0^{-\frac{1}{2}} v \right\rangle .$$

- By Lax-Milgram $\forall r \in X^{-1} \;\; \exists! e \in X^1$

$$B(e,v) = \langle r, v \rangle \,, \;\; \forall v \in X^1.$$

- For $r =$ r.h.s of (err) $\in X^{-1}$, $v = e \in X^1$

$$\left\| \mathcal{C}_1^{-\frac{1}{2}} \mathcal{A}^{-1} e \right\|^2 + \lambda \left\| \mathcal{C}_0^{-\frac{1}{2}} e \right\|^2 = \frac{1}{\sqrt{n}} \left\langle \mathcal{A}^{-1} \mathcal{C}_1^{-1} \xi, e \right\rangle - \lambda \left\langle \mathcal{C}_0^{-1} u^\dagger, e \right\rangle$$

Proof - Analysing the error equation

- Norm equivalence

$$\|e\|^2_{\beta-2\ell} + \lambda \|e\|^2_1 \leq c \left(\frac{1}{\sqrt{n}} \langle \mathcal{A}^{-1} \mathcal{C}_1^{-1} \xi, e \rangle - \lambda \langle \mathcal{C}_0^{-1} u^\dagger, e \rangle \right).$$

- Strategy: Interpolate norms on e on the r.h.s. between norms on e on the l.h.s. Recall $\eta_i = (1-\theta_i)(\beta - 2\ell) + \theta_i, \ \theta_i \in [0,1]$.

- Use norm equivalence, Cauchy-Schwarz, interpolation inequality, Cauchy with ε, Young's inequality

$$\|e\|^2_{\beta-2\ell} + \lambda \|e\|^2_1 \leq c \left(\frac{1}{\sqrt{n}} \left\langle \mathcal{C}_0^{\frac{\eta_1}{2}} \mathcal{A}^{-1} \mathcal{C}_1^{-1} \xi, \mathcal{C}_0^{-\frac{\eta_1}{2}} e \right\rangle - \lambda \left\langle \mathcal{C}_0^{\frac{\eta_2}{2}-1} u^\dagger, \mathcal{C}_0^{-\frac{\eta_2}{2}} e \right\rangle \right)$$

$$\leq c \left(\frac{1}{\sqrt{n}} \|\xi\|_{2\beta-2\ell-\eta_1} \|e\|_{\eta_1} + \lambda \left\| u^\dagger \right\|_{2-\eta_2} \|e\|_{\eta_2} \right)$$

Proof - Analysing the error equation

$$\leq c\left(\frac{1}{\sqrt{n}}\lambda^{-\frac{\theta_1}{2}}\|\xi\|_{2\beta-2\ell-\eta_1}\|e\|_{\beta-2\ell}^{1-\theta_1}(\lambda^{\frac{1}{2}}\|e\|_1)^{\theta_1}+\lambda^{1-\frac{\theta_2}{2}}\|u^\dagger\|_{2-\eta_2}\|e\|_{\beta-2\ell}^{1-\theta_2}(\lambda^{\frac{1}{2}}\|e\|_1)^{\theta_2}\right)$$

$$\leq \frac{c}{2\varepsilon}\left(\frac{1}{n}\lambda^{-\theta_1}\|\xi\|_{2\beta-2\ell-\eta_1}^2+\lambda^{2-\theta_2}\|u^\dagger\|_{2-\eta_2}^2\right)$$
$$+\frac{c\varepsilon}{2}\left(\|e\|_{\beta-2\ell}^{2-2\theta_1}(\lambda^{\frac{1}{2}}\|e\|_1)^{2\theta_1}+\|e\|_{\beta-2\ell}^{2-2\theta_2}(\lambda^{\frac{1}{2}}\|e\|_1)^{2\theta_2}\right)$$

$$\leq \frac{c}{2\varepsilon}\left(\frac{1}{n}\lambda^{-\theta_1}\|\xi\|_{2\beta-2\ell-\eta_1}^2+\lambda^{2-\theta_2}\|u^\dagger\|_{2-\eta_2}^2\right)$$
$$+\frac{c\varepsilon}{2}\left((2-\theta_1-\theta_2)\|e\|_{\beta-2\ell}^2+(\theta_1+\theta_2)\lambda\|e\|_1^2\right).$$

Proof - Analysing the error equation

- For ε small enough

$$\|e\|^2_{\beta-2\ell} + \lambda \|e\|^2_1 \leq c(\frac{1}{n}\lambda^{-\theta_1} \|\xi\|^2_{2\beta-2\ell-\eta_1} + \lambda^{2-\theta_2} \|u^\dagger\|^2_{2-\eta_2}).$$

- Then

$$\mathbb{E} \|e\|^2_{\beta-2\ell} \leq c\mathbb{E}(\kappa^2)(\frac{1}{n}\lambda^{-\theta_1} + \lambda^{2-\theta_2})$$

$$\mathbb{E} \|e\|^2_1 \leq c\mathbb{E}(\kappa^2)(\frac{1}{n}\lambda^{-1-\theta_1} + \lambda^{1-\theta_2}).$$

- Optimize choosing $\lambda = n^p$ and interpolate. $\square$

Synopsis

We have

- Considered a linear Inverse Problem in infinite dimensions;
- Adopted a Bayesian approach with Gaussian prior and Gaussian noise;
- Seen that the posterior is Gaussian with mean the minimizer of a Tikhonov-Phillips functional;
- Shown posterior consistency results in the small noise limit.

References - Further Reading

- S. Agapiou, S. Larsson and A. M. Stuart, *Posterior Consistency of the Bayesian Approach to Linear Ill-Posed Inverse Problems*, http://arxiv.org/abs/1203.5753
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