PDE Techniques for Bayesian Inverse Problems

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Outline

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Inverse Problems

- Inverse Problems are concerned with determining causes for a desired or an observed effect.
- $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ known self-adjoint, positive definite linear operator with bounded inverse, $\mathcal{X}$ separable Hilbert space.
- Consider the inverse problem to find u from y, where y noisy observation of $\mathcal{A}^{-1}u$.
- Model:

$$y = \mathcal{A}^{-1}u + \frac{1}{\sqrt{n}}\xi, \tag{1}$$

where $\frac{1}{\sqrt{n}}\xi$ an additive noise.

Inverse Problems are Ill-Posed - Deterministic Approach

- Problem (1) is in general ill-posed:
  - The existence of a solution is not guaranteed
  - Solutions do not depend continuously on the data
- Tikhonov-Philips Regularization: u approximated by the minimizer of

$$J_0(u) := \frac{1}{2}\left\|C_1^{-\frac{1}{2}}(y - \mathcal{A}^{-1}u)\right\|^2 + \frac{\lambda}{2}\left\|C_0^{-\frac{1}{2}}u\right\|^2,$$

where $C_i : \mathcal{X} \to \mathcal{X},\ i = 0, 1$ self-adjoint, possibly compact on $\mathcal{X}$, positive definite linear operators.

- λ called the regularization parameter, is chosen as appropriate function of noise level $n^{-\frac{1}{2}}$ which shrinks to zero as $n \to \infty$ to recover unknown u.

Bayesian Approach

Adopt a Bayesian approach to solve problem (1): assume $\xi \sim \mathcal{N}(0, C_1)$, where $C_1 : \mathcal{X} \to \mathcal{X}$ selfadjoint positive definite but not necessarily trace class linear operator (white noise allowed).

- Likelihood: for fixed u the law of $y|u$, called the *likelihood*, is $\mathcal{N}(\mathcal{A}^{-1}u, \frac{1}{n}C_1)$.
- Prior: we choose a *prior* distribution for the unknown u, based on prior knowledge about u. Let $u \sim \mathcal{N}(0, \tau^2 C_0)$, where $C_0 : \mathcal{X} \to \mathcal{X}$ selfadjoint positive definite trace class linear operator.
- Posterior: in the Bayesian Approach, the solution of problem (1) is the distribution of $u|y$, called the *posterior* distribution.

Bayesian Approach - Finite Dimensions

- Bayes rule:

$$\mathbb{P}(u|y) \propto \mathbb{P}(y|u)\mathbb{P}(u)$$

- In $\mathbb{R}^d$

$$\pi^y(u) \propto \rho(y - \mathcal{A}^{-1}u)\pi(u).$$

In general calculation of posterior difficult. Gaussian prior and noise case simpler.

- Assuming $\mathcal{X} = \mathbb{R}^d$, $\mathcal{A}$, C_1, $C_0 \in \mathbb{R}^{d\times d}$ and defining $\lambda = \frac{1}{n\tau^2}$ we have

$$\pi^y(u) \propto \exp\left(-nJ_0(u)\right)$$

$$nJ_0(u) = \frac{n}{2}\left\|C_1^{-\frac{1}{2}}(y - \mathcal{A}^{-1}u)\right\|^2 + \frac{1}{2\tau^2}\left\|C_0^{-\frac{1}{2}}u\right\|^2.$$

Bayesian Approach - Finite Dimensions

- This suggests that posterior is Gaussian $\mu^y = \mathcal{N}(m, \frac{1}{n}B_\lambda^{-1})$, where by completing the square

$$B_\lambda = \mathcal{A}^{-1}C_1^{-1}\mathcal{A}^{-1} + \lambda C_0^{-1} \tag{2}$$

$$B_\lambda m = \mathcal{A}^{-1}C_1^{-1}y. \tag{3}$$

- Observe B_λ depends on n and τ only through $\lambda = \frac{1}{n\tau^2}$.

- Observation: m is the minimizer of J hence also of J_0, that is the posterior mean is the Tikhonov-Philips solution of problem (1).

- We work in an infinite dimensional setting where formulae (2) and (3) can be justified (Theorem 1).

Bayesian Approach - Posterior Consistency

Assume observations

$$y^\dagger = \mathcal{A}^{-1}u^\dagger + \frac{1}{\sqrt{n}}\xi, \quad \xi \sim \mathcal{N}(0, C_1)$$

where $u^\dagger \in \mathcal{X}$ is the fixed true solution.

- This data model gives $\mu^{y^\dagger} = \mathcal{N}(m_\lambda^\dagger, \frac{1}{n}B_\lambda^{-1})$, where

$$B_\lambda m_\lambda^\dagger = \mathcal{A}^{-1}C_1^{-1}y^\dagger.$$

- We examine the consistency of the posterior mean in the sense

$$\mathbb{E}\left\|m_\lambda^\dagger - u^\dagger\right\|^2 \to 0,$$

as the noise disappears ($n \to \infty$) and for appropriate choice of scaling of prior $\tau = \tau(n)$ (Theorem 2). Assume a-priori known regularity of $u^\dagger$ and provide rates.

Outline

Hilbert Scale

- Recall we assume

$$y = \mathcal{A}^{-1}u + \frac{1}{\sqrt{n}}\xi$$

$u \sim \mu_0 = \mathcal{N}(0, \tau^2 C_0)$ and $\xi \sim \mathcal{N}(0, C_1)$ implying $y|u \sim \mathcal{N}(\mathcal{A}^{-1}u, \frac{1}{n}C_1)$.

- We work in the Hilbert Scale $(X^s)_{s\in\mathbb{R}}$, for $X^s := \overline{\mathcal{M}}^{\|\cdot\|_s}$ where

$$\mathcal{M} = \bigcap_{k=0}^{\infty} \mathcal{D}(C_0^{-k}),\ \langle u, v\rangle_s := \left\langle C_0^{-\frac{s}{2}}u, C_0^{-\frac{s}{2}}v\right\rangle \quad \text{and} \quad \|u\|_s := \left\|C_0^{-\frac{s}{2}}u\right\|.$$

Operator Interrelations - Prior Dominates

We have a number of assumptions on $\mathcal{A}$, C_0 and C_1 reflecting the idea

$$C_1 \simeq C_0^{\beta}$$

and

$$\mathcal{A}^{-1} \simeq C_0^{\ell},$$

for some $\beta \geq 0, \ell > 0$, where $\simeq$ means they induce equivalent norms (precise later).

- Let $\Delta = 2\ell - \beta + 1$. We have

  $$B_\lambda \simeq C_0^{2\ell-\beta} + \lambda C_0^{-1}$$

  thus the sign of Δ determines which term dominates. We require $\Delta > 0$ which means that the prior is regularizing.

Posterior Identification

Theorem 1

Let $y = \mathcal{A}^{-1}u + \frac{1}{\sqrt{n}}\xi$, where $u \sim \mathcal{N}(0, \tau^2 C_0)$ and $\xi \sim \mathcal{N}(0, C_1)$. Then

$$\mu^y = \mathcal{N}(m, \frac{1}{n}B_\lambda^{-1}),$$

where m and B_λ are given by

$$B_\lambda = \mathcal{A}^{-1}C_1^{-1}\mathcal{A}^{-1} + \lambda C_0^{-1} \tag{2}$$

and

$$B_\lambda m = \mathcal{A}^{-1}C_1^{-1}y. \tag{3}$$

Posterior Consistency - Convergence Rates

Theorem 2

For every $\theta \in [0,1]$ for $\mu = (1-\theta)(\beta - 2\ell) + \theta$, by choosing

$$\tau(n) = n^{\frac{\theta_2 - \theta_1 - 1}{4 - 2\theta_2 + 2\theta_1}},$$

we have

$$\mathbb{E}\left\|m_\lambda^\dagger - u^\dagger\right\|_\mu^2 \leq c\mathbb{E}(\kappa^2) n^{\frac{\theta + \theta_2 - 2}{\theta_1 - \theta_2 + 2}},$$

where

$$\kappa = \max\left\{\|\xi\|_{2\beta - 2\ell - \mu_1}, \left\|u^\dagger\right\|_{2-\mu_2}\right\} \quad \text{for} \quad \mu_i = (1-\theta_i)(\beta - 2\ell) + \theta_i, \ i = 1,2$$

and where $\theta_1, \theta_2 \in [0,1]$ are chosen so that $\mathbb{E}(\kappa^2) < \infty$.

- - higher θ stronger error-norm gives slower rates
  - regular noise/truth allows smaller θ_1, θ_2, faster rates
  - limitations because $\theta_1, \theta_2 \in [0,1]$
  - choice of θ_1, θ_2 independent of θ

Outline

Setting - Regularity of Prior

- Let $\{\lambda_k^2, \phi_k\}_{k\in\mathbb{N}}$ orthonormal eigenbasis of C_0 in $\mathcal{X}$. Since C_0 trace class $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$.

Assumption 1

Let $\sigma_0 \in (0,1]$ be such that $\sum_{k=1}^{\infty} \lambda_k^{2(1-\sigma)} < \infty \quad \forall \sigma < \sigma_0$.

eg. if $\sigma_0 = 1$, λ_k^2 decay faster than any negative power of k.

Lemma 1

Let ζ white noise wrt C_0. Then $\mathbb{E}\left\|C_0^{\frac{s}{2}}\zeta\right\|^2 < \infty$ for all $s > s_0 := 1 - \sigma_0$.

- Proof: ζ white noise thus $C_0^{\frac{s}{2}}\zeta \sim \mathcal{N}(0, C_0^s)$. To have $\mathbb{E}\left\|C_0^{\frac{s}{2}}\zeta\right\|^2 < \infty$, it suffices that C_0^s trace class, $\sum_{k=1}^{\infty} \lambda_k^{2s} < \infty$. By Assumption 1 need $s > 1 - \sigma_0$. $\square$

Assumptions - Norm Equivalence

Assumption 2

Suppose there exist $\beta \geq 0$, $\ell > 0$ and constants c_1-$c_6 > 0$ such that for $\Delta = 2\ell - \beta + 1$ we have

1. $\Delta > 2(1-\sigma_0)(> 0)$;
2. $\left\| C_1^{-\frac{1}{2}} \mathcal{A}^{-1} u \right\| \asymp \left\| C_0^{\ell - \frac{\beta}{2}} u \right\|, \ \forall u \in X^{\beta - 2\ell}$;
3. $\left\| C_0^{-\frac{\rho}{2}} C_1^{\frac{1}{2}} u \right\| \leq c_1 \left\| C_0^{\frac{\beta-\rho}{2}} u \right\|, \ \forall u \in X^{\rho-\beta}, \ \forall \rho < \beta + \sigma_0 - 1$;
4. $\left\| C_0^{\frac{s}{2}} C_1^{-\frac{1}{2}} u \right\| \leq c_2 \left\| C_0^{\frac{s-\beta}{2}} u \right\|, \ \forall u \in X^{\beta - s}, \ \forall s \in (s_0, 1]$;
5. $\left\| C_0^{-\frac{s}{2}} C_1^{-\frac{1}{2}} \mathcal{A}^{-1} u \right\| \leq c_3 \left\| C_0^{\frac{2\ell-\beta-s}{2}} u \right\|, \ \forall u \in X^{s+\beta-2\ell}, \ \forall s \in (s_0, 1]$;
6. $\left\| C_0^{\frac{\mu}{2}} \mathcal{A}^{-1} C_1^{-1} u \right\| \leq c_4 \left\| C_0^{\frac{\mu}{2}+\ell-\beta} u \right\|, \ \forall u \in X^{2\beta-2\ell-\mu}, \ \forall \mu \in [\beta - 2\ell, 1]$;
7. $\left\| \mathcal{A}^{-1} C_1^{-1} \mathcal{A}^{-1} u \right\| \leq c_5 \left\| C_0^{2\ell-\beta} u \right\|, \ \forall u \in X^{2\beta - 4\ell}$.

Making Sense of (3), $B_\lambda m = \mathcal{A}^{-1} C_1^{-1} y$

- Consider
$$B_\lambda w = r, \tag{4}$$
recalling $B_\lambda = \mathcal{A}^{-1} C_1^{-1} \mathcal{A}^{-1} + \lambda C_0^{-1}$.
- Let $r \in X^{-1}$. An element $z \in X^1$ is called a weak solution of (4), if $\forall v \in X^1$
$$\langle B_\lambda z, v \rangle = \langle r, v \rangle .$$

Proposition 1

Under the Assumptions 2, a unique weak solution of (4) exists.

- Proof: Apply Lax-Milgram since $B(u, v) := \langle B_\lambda u, v \rangle$ coercive and continuous bilinear form on $X^1 \times X^1$.$\square$

Additional Assumption

We make the following additional assumption needed to show that the posterior is Gaussian and justify the formulae (2) and (3).

Assumption 3

If $r \in X$ then the weak solution $w \in X^1$ of the equation (4) satisfies

$$\left\| \mathcal{A}^{-1} C_1^{-1} \mathcal{A}^{-1} w \right\| \leq c \left\| r \right\| ,$$

for some constant $c > 0$.

Outline

1. Introduction/Background
2. Statement of Main Results
3. Setting/Assumptions
4. Proof of Theorem 1
5. Proof of Theorem 2
6. Conclusions

Sketch of Proof of Theorem 1

Want to show for $y = \mathcal{A}^{-1}u + \frac{1}{\sqrt{n}}\xi$, where $u \sim \mathcal{N}(0, \tau^2 C_0)$, $\xi \sim \mathcal{N}(0, C_1)$ that $\mu^y = \mathcal{N}(m, \frac{1}{n}B_\lambda^{-1})$, where m, B_λ given by (3), (2).

- Characterize μ^y through Radon-Nikodym derivative wrt prior, μ_0:

$$\frac{d\mu^y}{d\mu_0}(u) = \frac{1}{Z(y)}\exp(-\Phi(u,y)),$$

$$\Phi(u,y) := \frac{n}{2}\left\| C_1^{-\frac{1}{2}}\mathcal{A}^{-1}u\right\|^2 - n\left\langle C_1^{-\frac{1}{2}}y, C_1^{-\frac{1}{2}}\mathcal{A}^{-1}u\right\rangle$$

  $Z(y)$ normalizing constant.
- This gives existence and Gaussianity of posterior.
- Finite dim approximation in Karhunen-Loeve basis of prior:

$$\frac{d\mu^{N,y}}{d\mu_0}(u) = \frac{1}{Z^N}\exp(-\Phi^N(u,y))$$

  $\Phi^N(u,y) = \Phi(P^N u; y)$, P^N projection onto $\mathcal{X}^N$:=span$\{\phi_1, \ldots, \phi_N\}$

Sketch of Proof of Theorem 1

- Lemma 1 gives $\mathcal{A}^{-1}C_1^{-1}y \in X^{-1}$ thus $\exists! m \in X^1$ s.t. $B_\lambda m = \mathcal{A}^{-1}C_1^{-1}y$.
- Let $C = \frac{1}{n}B_\lambda^{-1}$. Completing the square gives Galerkin Approximation of (3), (2): $\mu^{N,y} = \mathcal{N}(m^N, C^N)$ where

$$m^N = \begin{cases} \tilde{m}^N, & in \quad \mathcal{X}^N \\ 0, & in \quad (\mathcal{X}^N)^\perp \end{cases} \quad \text{and} \quad C^N = \begin{cases} \tilde{C}^N, & in \quad \mathcal{X}^N \\ C_0, & in \quad (\mathcal{X}^N)^\perp \end{cases},$$

$(\tilde{C}^N)^{-1} = P^N C^{-1} P^N$ and $\tilde{m}^N = n\tilde{C}^N P^N \mathcal{A}^{-1}C_1^{-1}y$.

- Pass to the limit
  - $\mu^{N,y} \Rightarrow \mu^y$
  - $\mu^{N,y} \Rightarrow \mathcal{N}(m, \frac{1}{n}B_\lambda^{-1})$.

  Uniqueness of weak limit completes the proof.

Characterization of Posterior Mean and Covariance

Proposition

i)

$$\lim_{N\to\infty} \left\| m^N - m \right\| = 0,$$

ii)

$$\lim_{N\to\infty} \left\| \sqrt{C^N} - \sqrt{C} \right\|_{HS} = 0.$$

i) Galerkin Orthogonality: $B(m - m^N, v) = 0,\ \forall v \in P^N X^1.$

For any $z \in P^N X^1$

$$c \left\| m - m^N \right\|_1^2 \le B(m - m^N, m - m^N) = B(m - m^N, m - z) \le c \left\| m - m^N \right\|_1 \left\| m - z \right\|_1$$

$$z = P^N m \quad \Rightarrow \quad \left\| m - m^N \right\|_1^2 \le c \left\| m - P^N m \right\|_1^2 = \sum_{k>N} \langle m, \phi_k \rangle^2 \lambda_k^{-2} \xrightarrow{N} 0,$$

since $m \in X^1$ thus $\sum_{k=1}^{\infty} \langle m, \phi_k \rangle^2 \lambda_k^{-2} < \infty$.

Posterior Justification

ii) We have

$$\left\|\sqrt{C^N}-\sqrt{C}\right\|_{HS}=\sum_{k>N}\left\|\left(\sqrt{C_0}-\sqrt{C}\right)\phi_k\right\|^2.$$

Let $r_k \in X^1, k \in \mathbb{N}$ be the unique weak solution of $C^{-1}r_k=\phi_k$. Then

$$\sum_{k>N}\left\|\left(\sqrt{C_0}-\sqrt{C}\right)\phi_k\right\|^2 \le \sum_{k>N}\left(\left\|\sqrt{C_0}\phi_k\right\|+\left\|\sqrt{C}\phi_k\right\|\right)^2$$

$$\le 2\left(\sum_{k>N}\langle C_0\phi_k,\phi_k\rangle+\sum_{k>N}\langle C\phi_k,\phi_k\rangle\right)=2\left(\sum_{k>N}\langle C_0\phi_k,\phi_k\rangle+\sum_{k>N}\langle r_k,\phi_k\rangle\right).$$

First sum goes to 0 as $N\to\infty$.

Posterior Justification

By definition of r_k, for $v_k = \tau^2 C_0 \phi_k \in X^1$

$$\sum_{k>N} \langle \phi_k, \tau^2 C_0 \phi_k \rangle = \sum_{k>N} \langle C^{-1} r_k, \tau^2 C_0 \phi_k \rangle$$

$$= \sum_{k>N} \langle n\mathcal{A}^{-1} C_1^{-1} \mathcal{A}^{-1} r_k, \tau^2 C_0 \phi_k \rangle + \sum_{k>N} \left\langle \frac{1}{\tau^2} C_0^{-1} r_k, \tau^2 C_0 \phi_k \right\rangle,$$

thus

$$\sum_{k>N} \langle r_k, \phi_k \rangle = \sum_{k>N} \langle \tau^2 C_0 \phi_k, \phi_k \rangle - \sum_{k>N} \langle n\tau^2 \mathcal{A}^{-1} C_1^{-1} \mathcal{A}^{-1} r_k, C_0 \phi_k \rangle$$

$$\leq \sum_{k>N} \left| \langle \tau^2 C_0 \phi_k, \phi_k \rangle \right| + \sum_{k>N} \left\| n\tau^2 \mathcal{A}^{-1} C_1^{-1} \mathcal{A}^{-1} r_k \right\| \left\| C_0 \phi_k \right\|$$

$$\leq \sum_{k>N} \left| \langle \tau^2 C_0 \phi_k, \phi_k \rangle \right| + \sum_{k>N} \left\| n\tau^2 \phi_k \right\| \left\| C_0 \phi_k \right\| \leq c \sum_{k>N} \left\| C_0 \phi_k \right\| \overset{N\to\infty}{\longrightarrow} 0,$$

where second inequality holds by Assumption 3 since $\phi_k \in X$. $\square$

Outline

Posterior Consistency - Convergence Rates

Recall we assume data $y^\dagger = \mathcal{A}^{-1}u^\dagger + \frac{1}{\sqrt{n}}\xi$ and want to estimate the error $m_\lambda^\dagger - u^\dagger$.

$$B_\lambda u^\dagger = \mathcal{A}^{-1}C_1^{-1}y^\dagger + \lambda C_0^{-1}u^\dagger - \frac{1}{\sqrt{n}}\mathcal{A}^{-1}C_1^{-1}\xi$$

$$B_\lambda m_\lambda^\dagger = \mathcal{A}^{-1}C_1^{-1}y^\dagger$$

therefore

$$m_\lambda^\dagger - u^\dagger = \frac{1}{\sqrt{n}}B_\lambda^{-1}\mathcal{A}^{-1}C_1^{-1}\xi - \lambda B_\lambda^{-1}C_0^{-1}u^\dagger.$$

Since ξ has zero mean, we have that for any $s \in \mathbb{R}$

$$\mathbb{E}\left\|m_\lambda^\dagger - u^\dagger\right\|_s^2 = \lambda^2\left\|B_\lambda^{-1}C_0^{-1}u^\dagger\right\|_s^2 + \frac{1}{n}\mathbb{E}\left\|B_\lambda^{-1}\mathcal{A}^{-1}C_1^{-1}\xi\right\|_s^2.$$

Operator Norm Bounds

Lemma 2

$$\left\|B_\lambda^{-1}\mathcal{A}^{-1}C_1^{-1}\right\|_{\mathcal{L}(X^{2\beta-2\ell-\mu_1},X^{\beta-2\ell})} \leq c\lambda^{-\frac{\theta_1}{2}}$$

and

$$\left\|B_\lambda^{-1}\mathcal{A}^{-1}C_1^{-1}\right\|_{\mathcal{L}(X^{2\beta-2\ell-\mu_1},X^{1})} \leq c\lambda^{-\frac{\theta_1+1}{2}},$$

for every $\theta_1 \in [0,1]$, where $\mu_1 = (1-\theta_1)(\beta-2\ell)+\theta_1$.

Lemma 3

$$\left\|B_\lambda^{-1}C_0^{-1}\right\|_{\mathcal{L}(X^{2-\mu_2},X^{\beta-2\ell})} \leq c\lambda^{-\frac{\theta_2}{2}}$$

and

$$\left\|B_\lambda^{-1}C_0^{-1}\right\|_{\mathcal{L}(X^{2-\mu_2},X^{1})} \leq c\lambda^{-\frac{\theta_2+1}{2}},$$

for every $\theta_2 \in [0,1]$, where $\mu_2 = (1-\theta_2)(\beta-2\ell)+\theta_2$.

Operator Norm Bounds

Proof of Lemma 3: Let $h \in X^{2-\mu_2}$. Then $r := C_0^{-1}h \in X^{-1}$. Indeed,

$$\left\| C_0^{-1}h \right\|_{-1} = \left\| C_0^{\frac{1}{2}} C_0^{-1}h \right\| = \|h\|_1 \leq c \, \|h\|_{2-\mu_2} ,$$

since $\mu_2 \leq 1$. Therefore, $\exists! z \in X^1$ weak solution of $B_\lambda w = r$, i.e. $\forall v \in X^1$

$$\langle B_\lambda z, v \rangle = \left\langle C_0^{-1}h, v \right\rangle .$$

Choose $v = z$

$$\left\langle \mathcal{A}^{-1} C_1^{-1} \mathcal{A}^{-1} z, z \right\rangle + \lambda \left\langle C_0^{-1} z, z \right\rangle = \left\langle C_0^{\frac{\mu_2}{2}-1} h, C_0^{-\frac{\mu_2}{2}} z \right\rangle$$

and by Assumption 2 and the Cauchy-Schwarz inequality on rhs, we get

$$\|z\|_{\beta-2\ell}^2 + \lambda \, \|z\|_1^2 \leq c \left\| C_0^{\frac{\mu_2}{2}-1} h \right\| \|z\|_{\mu_2} .$$

Operator Norm Bounds

Interpolate norm on z on the rhs between norms on z on the lhs, then use Cauchy with ε inequality and then Young's inequality for $p = \frac{1}{1-\theta_2}$, $q = \frac{1}{\theta_2}$ to get

$$\|z\|_{\beta-2\ell}^2 + \lambda \|z\|_1^2 \leq c \left\| C_0^{\frac{\mu_2}{2}-1} h \right\| \|z\|_{\beta-2\ell}^{1-\theta_2} \lambda^{-\frac{\theta_2}{2}} \left(\lambda^{\frac{1}{2}} \|z\|_1\right)^{\theta_2}$$

$$\leq \frac{c}{2\varepsilon}\left(\lambda^{-\theta_2} \left\| C_0^{\frac{\mu_2}{2}-1} h \right\|^2\right) + \frac{c\varepsilon}{2}\left(\|z\|_{\beta-2\ell}^{2(1-\theta_2)} \left(\lambda^{\frac{1}{2}} \|z\|_1\right)^{2\theta_2}\right)$$

$$\leq \frac{c}{2\varepsilon}\left(\lambda^{-\theta_2} \left\| C_0^{\frac{\mu_2}{2}-1} h \right\|^2\right) + \frac{c\varepsilon}{2}\left((1-\theta_2)\|z\|_{\beta-2\ell}^2 + \theta_2 \lambda \|z\|_1^2\right).$$

Choose $\varepsilon > 0$ small to get, for $c > 0$ independent of θ, λ,

$$\|z\|_{\beta-2\ell} \leq c\lambda^{-\frac{\theta_2}{2}} \left\| C_0^{\frac{\mu_2}{2}-1} h \right\| \quad \text{and} \quad \|z\|_1 \leq c\lambda^{-\frac{\theta_2+1}{2}} \left\| C_0^{\frac{\mu_2}{2}-1} h \right\|.$$

Replace $z = B_\lambda^{-1} C_0^{-1} h$ to get the result. □

Posterior Consistency - Convergence Rates

Proof of Theorem 2: We use the norm bounds to get

$$\mathbb{E}\left\|m_\lambda^\dagger - u^\dagger\right\|_{\beta-2\ell}^2 \leq c\mathbb{E}(\kappa^2)(\lambda^{2-\theta_2} + \frac{1}{n}\lambda^{-\theta_1})$$

and

$$\mathbb{E}\left\|m_\lambda^\dagger - u^\dagger\right\|_1^2 \leq c\mathbb{E}(\kappa^2)(\lambda^{1-\theta_2} + \frac{1}{n}\lambda^{-\theta_1-1}).$$

Optimize by choosing $\lambda = \lambda(n) = n^p$ such that $\lambda^{2-\theta_2} = \frac{1}{n}\lambda^{-\theta_1}$ i.e. $p = \frac{-1}{2-\theta_2+\theta_1}$ to obtain that

$$\mathbb{E}\left\|m_\lambda^\dagger - u^\dagger\right\|_{\beta-2\ell}^2 \leq c\mathbb{E}(\kappa^2)n^{\frac{\theta_2-2}{\theta_1-\theta_2+2}} \quad \text{and} \quad \mathbb{E}\left\|m_\lambda^\dagger - u^\dagger\right\|_1^2 \leq c\mathbb{E}(\kappa^2)n^{\frac{\theta_2-1}{\theta_1-\theta_2+2}}.$$

Interpolate between two last estimates to obtain the claimed rate.

□

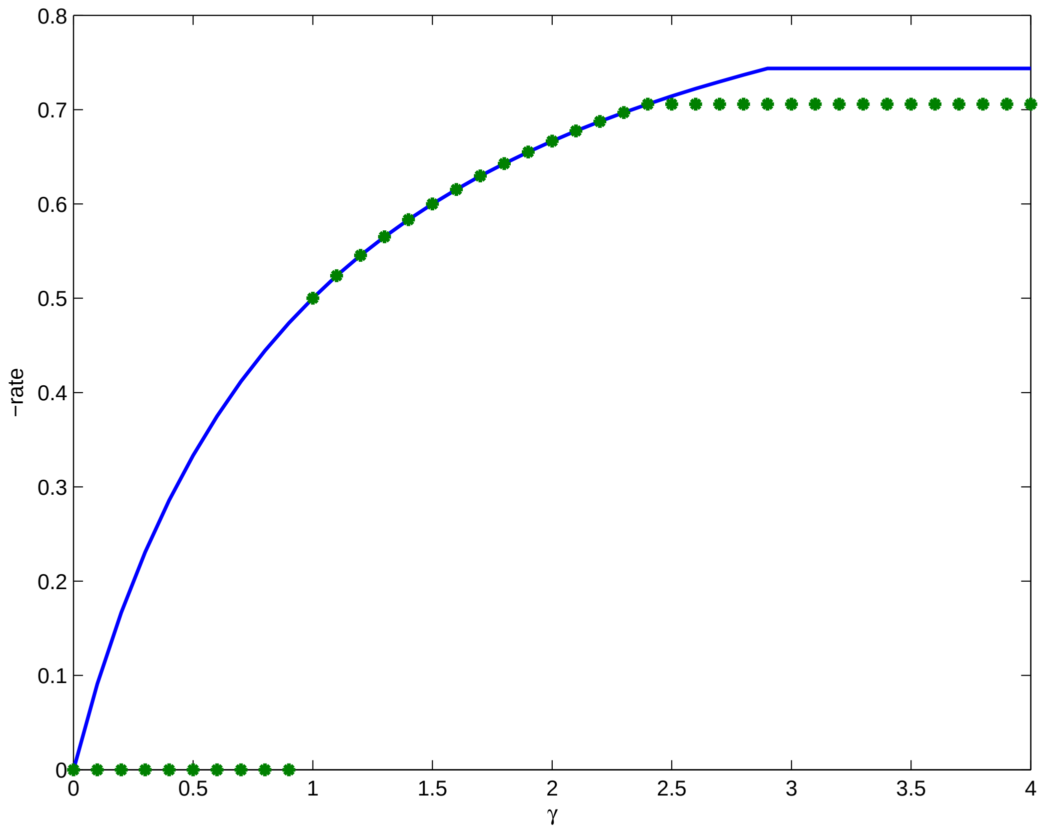
Outline

Optimality

Diagonal Case: $C_0 = (-\Delta)^{-1}$, $\mathcal{A}^{-1} = C_0^{\ell}$ and $C_1 = C_0^{\beta}$ gives sharp rates.

Our assumptions are satisfied trivially thus we can compare to our rates.

- We assume $u^{\dagger} \in X^{\gamma}$ and calculate the rate of convergence with both methods for $\ell = \beta = 1/2$.



Synopsis

We have

- Considered a linear Inverse Problem in infinite dimensions;
- Adopted a Bayesian approach with Gaussian prior and Gaussian noise;
- Identified the posterior as Gaussian and characterized its mean and covariance;
- Shown posterior consistency results in the small noise limit.

- Thank You!

References - Further Reading

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