Abstract

We study the numerical approximation to the solution of a fourth order singularly perturbed boundary value problem, by the Finite Element Method (FEM). In particular, we consider the $hp$ version of the FEM, in which the mesh size $h$ and the polynomial degree $p$ of the approximating polynomials change to improve accuracy. The solution to such problems typically exhibits boundary layers due to the presence of a small parameter multiplying the highest derivative. Moreover, given that the problem under consideration is of fourth order, the discrete problem is posed in a finite dimensional subspace of the usual Sobolev space $H^2$; for this reason we have to construct hierarchical $hp$ basis functions that are continuously differentiable, something that is novel. Under the assumption of sufficiently smooth input data, the proposed $hp$ FEM yields an extremely accurate approximation, provided the appropriate mesh-degree combination is used. The method is implemented in MATLAB and our numerical results confirm our conjecture: the method is robust (with respect to the singular perturbation parameter) and converges exponentially fast, as $p$ is increased, when the error is measured in the natural energy norm (associated with the variational problem). An outline of how this can be proved is also discussed.